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An Infinite thermoelastic Long Annular Cylinder with Variable Thermal Conductivity

A.A. El-Bary

Basic and Applied Science Department, Arab Academy of Science and Technology,
Po. Box 1029, Alexandria, Egypt.

Abstract: A model of theory of generalized thermoelasticity with one relaxation time will be constructed considering the thermal conductivity to be variable and will be applied to an infinitely long annular cylinder whose inner surface is traction free and subjected to a thermal and mechanical shocks, while the outer surface is traction free but thermally isolated. Laplace transform techniques are used and the solution in the Laplace transform domain is obtained by using a direct approach. The inverse of the Laplace transform is done numerically using a method based on Fourier expansion techniques. The results for the temperature increment, the stress components and the displacement component are illustrated graphically.

Key words: Elasticity, Thermoelasticity, Thermal Shock, Mechanical Shock

INTRODUCTION

The classical uncoupled theory of thermoelasticity predicts two phenomena not compatible with physical observations. First, the equation of heat conduction of this theory does not contain any elastic terms. Second, the heat equation is of a parabolic type, predicting infinite speeds of propagation for heat waves.

Biot[1] introduced the theory of coupled thermoelasticity to overcome the first shortcoming. The governing equations for this theory are coupled, eliminating the first paradox of the classical theory. However, both theories share the second shortcoming since the heat equation for the coupled theory is also parabolic.

Two generalizations to the coupled theory were introduced. The first is due to Lord and Shulman[2], who obtained a wave-type heat equation by postulating a new law of heat conduction to replace the classical Fourier's law. Since the heat equation of this theory is of the wave-type, it automatically ensures finite speeds of propagation for heat and elastic waves. The remaining governing equations for this theory, namely, the equations of motion and constitutive relations, remain the same as those for the coupled and the uncoupled theories.

The second generalization to the coupled theory of elasticity is what is known as the theory of thermoelasticity with two relaxation times or the theory of temperature-rate-dependent thermoelasticity. Müller[3], in a review of the thermodynamics of thermoelastic solids, proposed an entropy production inequality, with the help of which he considered restrictions on a class of constitutive equations. A generalization of this inequality was proposed by Green and Laws[4]. Green and Lindsay obtained an explicit version of the constitutive equations in[5]. These equations were also obtained independently by Suhubi[6]. This theory contains two constants that act as relaxation times and modify all the equations of the coupled theory, not only the heat equation. The classical Fourier's law of heat conduction is not violated if the medium under consideration has a center of symmetry. Eraby and Suhubi[7] studied wave propagation in a cylinder. Ignaczak[8] studied a strong discontinuity wave and obtained a decomposition theorem[9]. Ezzat[10] has also obtained the fundamental solution for this theory.

Modern structural elements are often subjected to temperature changes of such magnitude that their material properties may no longer be regarded as having constant values even in an approximate sense. The thermal and mechanical properties of materials vary with temperature, so that the temperature dependence of material properties must be taken into consideration in the thermal stress analysis of these elements[11,12].

Some problems have been solved by Sherief and. Megahed[14], Sherief and Kamal[15], El-Magrby and Youssef[16] , El-Bary and Youssef[17] .

In[18] Youssef and El-Bary discussed the thermal shock problem to the general case with variable thermal conductivity .

In this work, we shall construct a new model of theory of generalized thermoelasticity with one relaxation time considering the thermal conductivity to be variable. We consider an infinitely long annular cylinder whose inner surface is traction free and subjected to a thermal shock, the outer surface is also traction free but thermally isolated and the medium parameters quiescent initial state.

Corresponding Author: A.A. El-Bary, Basic and Applied Science Department, Arab Academy of Science and Technology, Po. Box 1029, Alexandria, Egypt.

Laplace transform techniques are used. The inverses Laplace transforms are obtained numerically.

The Governing equations:
The heat equation[11]:

$$\left(K\theta_{,i}\right)_{,i} = \left(1+\tau_o\frac{\partial}{\partial t}\right)\left[\rho C_E\dot{\theta}+\gamma T_o\dot{e}\right] \quad i=1,2,3 \tag{1}$$

where, and

$$\left|T-T_o\right|=\theta \quad K=K(\theta)=K_o(1+K_1\theta) \text{ and } \quad \rho C_E=\frac{K}{k}$$

K is called the thermal conductivity (K_1 is a small value) and *k* is the diffusivity (assumed to be constant).

we get. $$\left(K\theta_{,i}\right)_{,i} = \left(1+\tau_o\frac{\partial}{\partial t}\right)\left[\frac{K}{k}\dot{\theta}+\gamma T_o\dot{e}\right] \tag{2}$$

let the mapping $J=\frac{1}{K_o}\int_0^{\theta}K(\theta')d\theta'$ [11],

then, we have $K_oJ_{,ii}=\left[K(\theta)\theta_{,i}\right]_{,i}$ in addition $K_o\dot{J}=K(\theta)\dot{\theta}$
Hence, the heat equation will take the form

$$J_{,ii}=\left[\frac{\partial}{\partial t}+\tau_o\frac{\partial^2}{\partial t^2}\right]\left[\frac{J}{k}+\frac{\gamma T_o}{K_o}e\right] \quad . \tag{3}$$

The equations of motion [11]

$$\rho\ddot{u}_i=(\lambda+\mu)u_{j,ji}+\mu u_{i,jj}-\gamma\theta_{,i}+\rho F_i, \tag{4}$$

where $J=\theta+\frac{K_1}{2}\theta^2$ and $J_{,i}=\theta_{,i}(1+K_1\theta)$,

Then, we have,
neglecting the small values and the body forces, we get

$$\rho\ddot{u}_i=(\lambda+\mu)u_{j,ji}+\mu u_{i,jj}-\gamma\frac{J_{,i}}{(1+K_1\theta)}+\rho F_i \tag{5}$$

The constitutive relation

$$\sigma_{ij}=2\mu e_{ij}+(\lambda e_{kk}-\gamma\theta)\delta_{ij}, \tag{6}$$

where ? is the density, t is the time, T is the absolute temperature, ? , μ are Lame's constants, τ_o is the relaxation time, C^E is the specific heat at constant deformation and $\gamma=(3\lambda+2\mu)\alpha_t$ where a_t is the coefficient of linear thermal expansion.

Formulation of the Problem: we consider an infinitely long annular cylinder whose inner surface is traction free and subjected to a thermal shock, while the outer surface also is traction free but thermally isolated. we assume also that there are no external body forces or heat sources acting in the medium.

we use a cylindrical system of coordinates (r,y,z) with the z-axis lying along the axis of the cylinder.

Due to symmetry, the problem is one-dimensional with all the functions considered depending on the radial distance r and the time t.

The displacement vector has the components

$$u_r=u(r,t), \quad u_\psi(r,t)=u_z(r,t)=0 \tag{7}$$

From equation (3), the heat equation takes the form

$$\nabla^2J=\left[\frac{\partial}{\partial t}+\tau_o\frac{\partial^2}{\partial t^2}\right]\left[\frac{J}{k}+\frac{\gamma T_o}{K_o}e\right], \tag{8}$$

where $\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}$,

from equation (5), the equation of motion has the form

$$\rho\ddot{u}=(\lambda+2\mu)\frac{\partial e}{\partial r}-\gamma\frac{\partial J}{\partial r}, \tag{9}$$

where $$e=\frac{1}{r}\frac{\partial(ru)}{\partial r}, \tag{10}$$

and fromequation (6), the constitutive equations take the forms

$$\sigma_{rr}=2\mu\frac{\partial u}{\partial r}+\lambda e-\gamma J, \tag{11}$$

$$\sigma_{\psi\psi}=2\mu\frac{u}{r}+\lambda e-\gamma J, \tag{12}$$

$$\sigma_{zz}=\lambda e-\gamma J, \tag{13}$$

$$\sigma_{zr}=\sigma_{\psi r}=\sigma_{zz}=0. \tag{14}$$

we will use the following non-dimensional variables

$$r'=\frac{cr}{k}, \quad u'=\frac{cu}{k}, \quad J'=\frac{J}{T_o} \quad t'=\frac{c^2t}{k} \quad \tau_o'=\frac{c^2\tau_o}{k} \quad \sigma'=\frac{\sigma}{\mu}$$

$$\theta'=\frac{k\theta}{cK_oT_o}, \quad R'=\frac{cR}{k}$$

where $g=\frac{\gamma k}{K_o}$, $\beta=\left(\frac{\lambda+2\mu}{\mu}\right)^{1/2}$, $c=\left(\frac{\lambda+2\mu}{\rho}\right)^{1/2}$

and $a=\frac{gT_o}{\mu\beta^2}$.

Using these non-dimensional variables, the above equations take the form (dropping the primes for convenience)

$$\nabla^2J=\left[\frac{\partial}{\partial t}+\tau_o\frac{\partial^2}{\partial t^2}\right]\left[J+ge\right] \tag{15}$$

$$\nabla^2e-a\nabla^2J=\frac{\partial^2e}{\partial t^2}, \tag{16}$$

$$\sigma_{rr}=\beta^2\frac{\partial u}{\partial r}+(\beta^2-2)\frac{u}{r}-bJ \tag{17}$$

$$\sigma_{\psi\psi}=(\beta^2-2)\frac{\partial u}{\partial r}+\beta^2\frac{u}{r}-bJ, \tag{18}$$

$$s_{zz} = \left(b^2 - 2\right)e - bJ, \qquad (19)$$

$$e = \frac{1}{r}\frac{\P\,(ru)}{\P\, r}. \qquad (20)$$

Formulation in the Laplace Transforms Domain: Taking the Laplace transform of both sides of equations (15)-(19), we obtain

$$\left[\nabla^2 - \left(s + t_o s^2\right)\right]\overline{J} = g\left(s + t_o s^2\right)\overline{e} \qquad (21)$$

$$\left(\nabla^2 - s^2\right)\overline{e} = a\nabla^2\overline{J}, \qquad (22)$$

$$\overline{s}_{rr} = b^2\frac{\P\,\overline{u}}{\P\, r} + \left(b^2 - 2\right)\frac{\overline{u}}{r} - b\overline{J}, \qquad (23)$$

$$\overline{s}_{yy} = \left(b^2 - 2\right)\frac{\P\,\overline{u}}{\P\, r} + b^2\frac{\overline{u}}{r} - b\overline{J}, \qquad (24)$$

$$\overline{s}_{zz} = \left(b^2 - 2\right)\overline{e} - b\overline{J}. \qquad (25)$$

Eliminating $\overline{e}$ from equations (15) and (16), we get

$$\left[\nabla^4 - \left[s^2 + (1+e)\left(s + t_o s^2\right)\right]\nabla^2 + s^2\left(s + t_o s^2\right)\right]\overline{J} = 0 \qquad (26)$$

where $e = ag$

In a similar manner, we can show that $\overline{e}$ satisfies the equation

$$\left[\nabla^4 - \left[s^2 + (1+e)\left(s + t_o s^2\right)\right]\nabla^2 + s^2\left(s + t_o s^2\right)\right]\overline{e} = 0, \qquad (27)$$

The solutions of equations (20) and (21) bounded at infinity can be written in the form

$$\overline{J} = \sum_{i=1}^{2} A_i\left(p_i^2 - s^2\right)K_0\left(p_i r\right) + B_i\left(p_i^2 - s^2\right)I_0\left(p_i r\right) \qquad (28)$$

where $K_o(.)$ and $I_o(.)$ are the modified Bessel functions of the first and the second kinds of order zero. A_1, A_2, B_1 and B_2 are all parameters depending on the parameter s of the Laplace transform p_1^2 and p_2^2 are the roots of the characteristic equation

$$p^4 - \left[s^2 + (1+e)\left(s + t_o s^2\right)\right]p^2 + s^2\left(s + t_o s^2\right) = 0. \qquad (29)$$

Using equation (22), we obtain

$$\overline{e} = a\sum_{i=1}^{2} A_i\, p_i^2\, K_0\left(p_i r\right) + B_i\, p_i^2\, I_0\left(p_i r\right) \qquad (30)$$

Substituting from equation (30) into the Laplace transform of equation (20), we obtain

$$\overline{u} = a\sum_{i=1}^{2} -A_i\, p_i\, K_1\left(p_i r\right) + B_i\, p_i\, I_1\left(p_i r\right) \qquad (31)$$

where $K_1(.)$ and $I_1(.)$ are the modified Bessel function of the first and the second kinds of order one.

In deriving equation (31), we have used the following well-known relation of the Bessel function

$$\int zK_o(z)\,dz = -z\,K_1(z),$$

Finally, substituting from equations (28), (30) and (31) into equations (23)-(25), we obtain the stress components in the form

$$\overline{s}_{rr} = a\sum_{i=1}^{2} A_i\left[b^2 s^2 K_0\left(p_i r\right) + \frac{2p_i}{r}K_1\left(p_i r\right)\right] + B_i\left[b^2 s^2 I_0\left(p_i r\right) - \frac{2p_i}{r}I_1\left(p_i r\right)\right] \qquad (32)$$

$$\overline{s}_{yy} = a\sum_{i=1}^{2} A_i\left[\left(b^2 s^2 - p_i^2\right)K_0\left(p_i r\right) - \frac{2p_i}{r}K_1\left(p_i r\right)\right] + B_i\left[\left(b^2 s^2 - p_i^2\right)I_0\left(p_i r\right) - \frac{2p_i}{r}I_1\left(p_i r\right)\right] \qquad (33)$$

$$\overline{s}_{zz} = a\sum_{i=1}^{2} A_i\left(b^2 s^2 - 2\right)p_i^2 K_0\left(p_i r\right) + B_i\left(b^2 s^2 - 2\right)p_i^2 I_0\left(p_i r\right) \qquad (34)$$

In order to evaluate the unknown parameters we will use the boundary conditions on the internal surface, $r = R_1$ and the outer surface, $r = R_2$ which are given by

(1)- Thermal boundary conditions

I- The internal surface $r = R_1$ is subjected to a thermal shock in the form $q(R,t) = q_o\,H(t)$ hence, we have $\overline{J}\left(R_1, s\right) = \frac{d}{s}$, (35)

where $d = \left(1 + \frac{K_1}{2}q_o\right)q_o$.

II- The outer surface $r = R_2$ is thermally isolated, i.e. at $r = R_2$ we have not any heat flux.

we will use the generalized Fourier law of heat conduction in non-dimensional form, namely

$$q_r + t_o\frac{\partial q_r}{\partial t} = -\frac{\partial J}{\partial r},$$

taking Laplace transform, we obtain $\overline{q}_r = -\frac{1}{1 + t_o s}\frac{\partial\overline{J}}{\partial r}$.

Now, from the condition at $r = R_2$ I have $\overline{q}_r = 0$.

Finally, we get $\frac{\partial\overline{J}\left(R_2, s\right)}{\partial r} = 0$. (36)

(2)- Mechanical boundary conditions

The internal and the outer surfaces $r = R_1$ and $r = R_2$ is traction free i.e.

$$\overline{s}_{rr}\left(R_1, s\right) = 0, \qquad (37)$$

$$\overline{s}_{rr}\left(R_2, s\right) = 0 \qquad (38)$$

Using equations (28), (32) and equations (35)-(38), we obtain the following system of linear algebraic equations in the unknown parameters A_1, A_2, B_1 and B_2

$$\sum_{i=1}^{2} A_i\left(p_i^2-s^2\right)K_0\left(p_iR_1\right)+B_i\left(p_i^2-s^2\right)I_0\left(p_iR_1\right)=\frac{d}{s}, \tag{39}$$

$$\sum_{i=1}^{2} A_i\left[b^2 s^2K_0\left(p_iR_1\right)+\frac{2p_i}{R_1}K_1\left(p_iR_1\right)\right] + B_i\left[b^2 s^2I_0\left(p_iR_1\right)-\frac{2p_i}{R_1}I_1\left(p_iR_1\right)\right]=0 \tag{40}$$

$$\sum_{i=1}^{2} -A_i\left(p_i^2-s^2\right)p_iK_1\left(p_iR_2\right)+B_i\left(p_i^2-s^2\right)p_iI_1\left(p_iR_2\right)=0, \tag{41}$$

$$\sum_{i=1}^{2} A_i\left[b^2 s^2K_0\left(p_iR_2\right)+\frac{2p_i}{R_2}K_1\left(p_iR_2\right)\right] + B_i\left[b^2 s^2I_0\left(p_iR_2\right)-\frac{2p_i}{R_2}I_1\left(p_iR_2\right)\right]=0 \tag{42}$$

Equations (39)-(42) constitute a system of four linear equations in the four unknown parameters A_i and B_i i=1,2, whose solution complete the solution of the problem in the Laplace transform domain.

After obtaining J, the temperature increment q can be obtained by solving equation (8) to give

$$q=\frac{-1+\sqrt{1+2K_1J}}{K_1}.$$

Inversion of the Laplace Transform: In order to invert the Laplace transform, we adopt a numerical inversion method based on a Fourier series expansion[13]

By this method the inverse of f(t) the Laplace transform $\bar{f}(s)$ is approximated by

$$f(t)=\frac{e^{ct}}{t_1}\left[\frac{1}{2}\bar{f}(c)+R1\sum_{k=1}^{N}\bar{f}\left(c+\frac{ikp}{t_1}\right)\exp\left(\frac{ikpt}{t_1}\right)\right],\ 0<t_1<2t,$$

where N is a sufficiently large integer representing the number of terms in the truncated Fourier series, chosen such that

$$\exp(ct)\,R1\left[\bar{f}\left(c+\frac{iNp}{t_1}\right)\exp\left(\frac{iNpt}{t_1}\right)\right]\le e_1,$$

where e_1 is a prescribed small positive number that corresponds to the degree of accuracy required. The parameter c is a positive free parameter that must be greater than the real part of all the singularities of $\bar{f}(s)$. The optimal choice of c was obtained according to the criteria described in[13].

Numerical Results: The copper material was chosen for purposes of numerical evaluations. The constants of the material were taken as follows:

$$K_o=386,\ a_T=1.78(10)^{-5},\ c_E=3.83.1,\ m=3.86(10)^{10}$$

$$l=7.76(10)^{10},\ r=8954,\ b^2=4,\ T_o=293,\ b=0.042$$

$$g=1.61,\ t_o=0.02,\ R_1=1,\ R_2=1,\ q_o=1,\ K_1=-0.1$$

$$a=0.0105$$

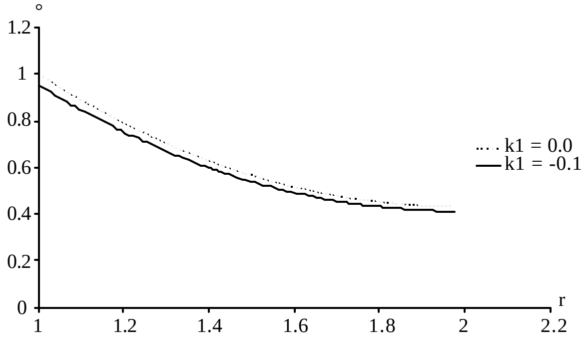

Fig. 1: The temperature distribution.

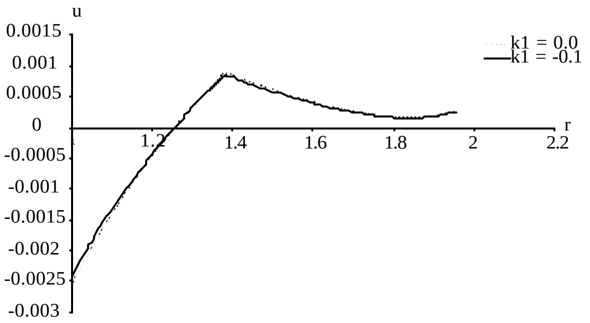

Fig. 2: The displacement distribution.

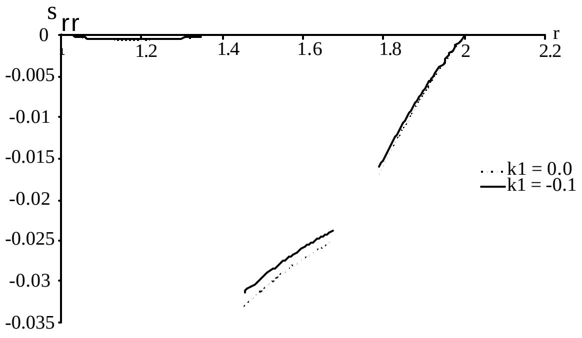

Fig. 3: The radial stress distribution.

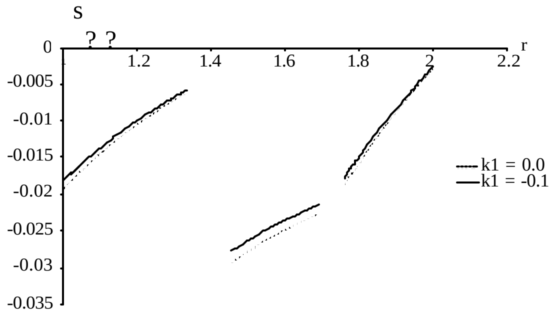

Fig. 4: The transeverse stress distribution.

The computations were carried out for one value of time, namely for t = 0.4. The results are illustrated graphically in figures (2), (3), (4), and (5) for the temperature increment q, the radial stress component s_{rr}, the transverse stress component s_{yy} and the radial displacement component u**,** respectively.

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